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The survey of preconditioners used for accelerating the rate of convergence in the Gauss–Seidel method

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Abstract

Several preconditioned iterative methods reported in the literature have been used for improving the convergence rate of the Gauss–Seidel method. In this article, on the basis of nonnegative matrix, comparisons between some splittings for such preconditioned matrices are derived. Simple numerical examples are also given.

Keywords: Preconditioning; Gauss–Seidel method; Finite difference; Laplace equation

1. Introduction and preliminaries

For the linear system

$$Ax = b, \quad A = (a_{ij}) \in \mathbb{R}^{n\times n}, \; x, b \in \mathbb{R}^n, \tag{1}$$

we consider left preconditioned linear system

$$PAx = Pb. \tag{2}$$

Here, $P \in \mathbb{R}^{n\times n}$ is nonsingular. For simplicity, we assume that A has unit diagonal entries and let $A = I - L - U$, where $-L$ and $-U$ are strictly lower and upper triangular parts of A, respectively.

Here the Gauss–Seidel method should be defined. Since PA may be a nonsymmetric matrix, it is necessary to consider two version of Gauss–Seidel method (forward and backward) for which the

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spectral radii of iteration matrices may have different value [11]. In this paper, we adopt only the forward Gauss–Seidel method.

Milaszewicz [7] considered the preconditioner

$$P_C = (I + C) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -a_{21} & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -a_{n1} & 0 & \cdots & \cdots & 1 \end{pmatrix} \tag{3}$$

to eliminate the elements of the first column below the diagonal of A. Then $A_C = (I + C)A$ can be written as follows:

$$A_C = (I + C)A = I - L - U + C - CU = M_C - N_C,$$

where

$$M_C = (I - D_C) - (L - C + E_C), \quad \text{and} \quad N_C = U + F_C$$

and D_C, E_C and F_C are the diagonal, strictly lower and strictly upper triangular parts of CU, respectively.

If M_C^{-1} is nonsingular, then the iteration matrix in the forward Gauss–Seidel method is defined by

$$\mathscr{L}_C^{\mathrm{f}} = M_C^{-1} N_C = \{(I - D_C) - (L - C + E_C)\}^{-1}(U + F_C).$$

The preconditioner P_S of Gunawardena et al. [3] eliminates the elements of the first upper codiagonal of A, where P_S is

$$P_S = (I + S) = \begin{pmatrix} 1 & -a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -a_{23} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -a_{n-1n} \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}. \tag{4}$$

Then $A_S = (I + S)A$ can be written as follows:

$$A_S = (I + S)A = I - L - U + S - SL - SU = M_S - N_S,$$

where

$$M_S = (I - D_S) - (L + E_S), \quad \text{and} \quad N_S = U - S + SU$$

and D_S, E_S are the diagonal and strictly lower triangular parts of SL, respectively. If M_S is nonsingular, then the Gauss–Seidel iteration matrix is defined by

$$\mathscr{L}_S^{\mathrm{f}} = M_S^{-1} N_S = \{(I - D_S) - (L + E_S)\}^{-1}(U - S + SU).$$

In 1997, Kohno et al. [4] proposed to use

$$P_\alpha = (I + \alpha S),$$

instead of $P_S=(I+S)$, where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})^{\mathrm{T}} \in \mathbb{R}^{n-1}$, $\alpha_i > 1$, $1 < i < n$. There exists $\alpha' \in \mathbb{R}^{n-1}$ such that A_α is a diagonally dominant matrix. (*Note*: The interested reader is referred to [4].) Then for $\alpha' > \alpha > 1$, $A_\alpha = (I + \alpha S)A$ can be written as follows:

$$A_\alpha = I - L - U + \alpha S - \alpha SL - \alpha SU = M_\alpha - N_\alpha,$$

where

$$M_\alpha = (I - \alpha D_S) - (L - \alpha E_S) \quad \text{and} \quad N_\alpha = (U - \alpha S + \alpha SU).$$

The elements of the preconditioned matrix $P_\alpha A$ is expressed as follows:

$$\tilde{a}_{ij} = a_{ij} - \alpha_i a_{ii+1} a_{i+1j}, \quad 1 \leqslant i < n, \; j = 1,2,\ldots,n. \tag{5}$$

Whenever $\alpha_i a_{ii+1} a_{i+1i} \neq 1$ for $1 \leqslant i < n$, M_α^{-1} exists and hence the Gauss–Seidel iteration matrix is defined by

$$\mathscr{L}_\alpha^{\mathrm{f}} = M_\alpha^{-1} N_\alpha = \{(I - \alpha D_S) - (L - \alpha E_S)\}^{-1}(U - \alpha S + \alpha SU).$$

Remark 1.1. (i) In the case $0 < \alpha_i \leqslant 1$. From (5) clearly $N_{\alpha<1} = U - \alpha S + \alpha SU \geqslant O$.

(ii) In the case $\alpha_i > 1$. The first upper codiagonal part of the preconditioned matrix is expressed as follows:

$$\begin{aligned}\tilde{a}_{ii+1} &= a_{ii+1} - \alpha_i a_{ii+1} a_{i+1i+1} \\ &= (1-\alpha_i) a_{ii+1} > 0, \quad \text{for} \quad i = 1,2,\ldots,n-1.\end{aligned}$$

Thus, $N_{\alpha>1}$ is not a nonnegative matrix. Write $M_{\alpha>1}^{-1} = (m_{ij})$, $N_{\alpha>1} = (n_{ij})$ and $T_{\alpha>1} = (t_{ij})$. As described after, $M_{\alpha>1}^{-1} = (m_{ij}) \geqslant O$, where $(m_{ij}) = 0$, for $i < j$. Then, for $\alpha_1 > 1$ the 2nd column of $T_{\alpha>1}$ has a negative value as follows:

$$\begin{aligned}t_{i2} &= m_{i1} n_{12} = m_{i1}(-\tilde{a}_{12}) \\ &= m_{i1}\{-(1-\alpha_1)a_{12}\} < 0, \quad 1 \leqslant i \leqslant n.\end{aligned}$$

Thus, $T_{\alpha>1}$ is not nonnegative.

By putting $\alpha a_{12} = a_{12}$, we have $t_{i2} = 0$, $1 \leqslant i \leqslant n$, that is, $M_\alpha^{-1} N_\alpha \geqslant O$. Moreover, obtaining optimum parameters α_i $(1 < i \leqslant n-1)$ are very difficult work. Thus, throughout the paper, we put as follows:

$$P_\alpha = \begin{pmatrix} 1 & -a_{12} & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\alpha a_{23} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 1 & -\alpha a_{n-1n} \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$

Then $A_{\alpha>1} = M_{\alpha>1} - N_{\alpha>1}$ is a weak regular splitting.

Moreover, Kotakemori et al. [6] considered the preconditioner

$$P_{\mathrm{U}} = (I + \beta U),$$

where U is a strictly upper triangular part of $-A$ and $\beta \geqslant 1$.

Recently, Kotakemori et al. [5] proposed to use

$$P_{\max} = (I + S_{\max}),$$

where $S_{\max}$ is defined by

$$S_{\max} = (s_{ij}) = \begin{cases} -a_{ik_i} & \text{for } 1 \leqslant i < n,\ i+1 \leqslant j \leqslant n, \\ 0 & \text{for otherwise}, \end{cases}$$

where $k_i = \min I_i$, $I_i = \{j : |a_{ij}| \text{ is maximal for } i+1 \leqslant j \leqslant n\}$ for $1 \leqslant i < n$. Then $A_{\max} = (I + S_{\max})A$ can be written as follows:

$$A_{\max} = (I + S_{\max})A = I - L - U + S_{\max} - S_{\max}L - S_{\max}U = M_{\max} - N_{\max},$$

where

$$M_{\max} = (I - D_{\max}) - (L + E_{\max}), \quad \text{and} \quad N_{\max} = U - S_{\max} + F_{\max} + S_{\max}U$$

and $D_{\max}$, $E_{\max}$ and $F_{\max}$ are the diagonal, strictly lower and strictly upper triangular parts of $S_{\max}L$, respectively.

If $M_{\max}$ is nonsingular, then the Gauss–Seidel iteration matrix is defined by

$$\mathscr{L}^{\mathrm{f}}_{\max} = M_{\max}^{-1} N_{\max} = \{(I - D_{\max}) - (L + E_{\max})\}^{-1}(U - S_{\max} + F_{\max} + S_{\max}U).$$

For the preconditioners $I + \alpha S$ and $I + \beta U$, a preconditioned effect does not appear on the last row of the matrix PA. Recently, Niki and Kohno proposed [9] to use a preconditioner

$$P_R = (I + S + R) = \begin{pmatrix} 1 & -a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -a_{23} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -a_{n-1n} \\ -a_{n1} & -a_{n2} & \cdots & -a_{nn-1} & 1 \end{pmatrix}. \tag{6}$$

Then A_R can be written as follows:

$$\begin{aligned} A_R &= (I + S + R)A \\ &= I - L - U + S - SL - SU + R - RL - RU = M_R - N_R, \end{aligned}$$

where

$$M_R = (I - D_S - D_R) - (L - R + RL + E_S + E_R), \quad N_R = U - S + SU$$

and D_R, E_R are the diagonal and strictly lower triangular parts of RU, respectively. If M_R is nonsingular, then the Gauss–Seidel iteration matrix is defined by

$$\mathscr{L}^{\mathrm{f}}_R = M_R^{-1} N_R = \{(I - D_S - D_R) - (L - R + RL + E_S + E_R)\}^{-1}(U - S + SU).$$

Remark 1.2. (i) it is easily obtained from (5) that $N \geqslant N_{0<\alpha<1} \geqslant N_{\alpha=1} \geqslant N_{1<\alpha\leqslant\alpha'}$, where $N_{1<\alpha\leqslant\alpha'}$ is not nonnegative.

(ii) For Gauss–Seidel splittings A_S and A_R, the following relation holds:
$N_S = N_R = U - S + SU$.

We review some known results used in Section 2.

Definition 1.3 (Neumann and Plemmons [8, Definition 3.22]). A real $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \leqslant 0$ for all $i \neq j$ is an M-matrix if A is nonsingular and A^{-1}.

Definition 1.4 (Axelsson [1, Definition 4.3]). The associated direct graph $G(A)$ of an $n \times n$ matrix A, consists of n vertices $P_1, P_2, \ldots, P_n$, where an edge leads from P_i to P_j if and only if $a_{ij} \neq 0$. A directed graph G is strongly connected if any ordered pair (P_i, P_j) of vertices G, there exists a sequence of edges (a path) which leads from P_i to P_j.

Definition 1.5 (Frommer and Szyld [2, Definition 3.3]). Let A be a real matrix. The representation $A = M - N$ is called

(i) regular if $M^{-1} \geqslant O$ and $N \geqslant O$,
(ii) weak regular if $M^{-1} \geqslant O$ and $M^{-1}N \geqslant O$,
(iii) M-splitting if M is an M-matrix and $N \geqslant O$,
(iv) H-splitting if $\langle M \rangle - |N|$ is an M-matrix,
(v) H-compatible splitting if $\langle A \rangle = \langle M \rangle - |N|$.

Theorem 1.6 (Frommer and Szyld [2, Theorem 3.4]). *Let $A = M - N$ be a splitting.*

(i) *If the splitting is regular or weak regular, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geqslant O$.*
(ii) *If the splitting is an M-splitting, then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.*
(iii) *If the splitting is an H-splitting, then A and M are H-matrices and $\rho(M^{-1}N) \leqslant \rho(\langle M \rangle^{-1}|N|) < 1$.*
(iv) *If the splitting is an M-splitting, then it is a regular splitting.*
(v) *If the splitting is an M-splitting and A is an M-matrix, then it is an H-splitting and also an H-compatible splitting.*
(vi) *If the splitting is an H-compatible splitting and A is an H-matrix, then it is an H-splitting and thus convergent.*

Definition 1.7. We call $A = M - N$ the Gauss–Seidel splitting of A, if $M = (D - E)$ and $N = F$, where D is the diagonal parts and $-E$ and $-F$ are strictly lower and upper triangular parts of A, respectively. In addition, the splitting is called

(i) Gauss–Seidel convergent if $\rho(M^{-1}N) < 1$,
(ii) Gauss–Seidel regular if $M^{-1} = (D - E)^{-1} \geqslant O$ and $N = F \geqslant O$,
(iii) Gauss–Seidel weak regular if $M^{-1} \geqslant O$ and $M^{-1}N \geqslant O$.

Theorem 1.8 (Axelsson [1, Lemma 6.1]). *A is monotone if and only if A is nonsingular with $A^{-1} \geqslant O$. Here a real matrix A is called monotone if $A\boldsymbol{x} \geqslant \mathbf{0}$ implies $\boldsymbol{x} \geqslant \mathbf{0}$.*

Theorem 1.9 (Varga [10, Theorem 2.20]). *Let $A \geqslant O$ be an $n \times n$ matrix. Then,*

(i) *A has a nonnegative real eigenvalue equal to its spectral radius.*
(ii) *To $\rho(A)$, there corresponds a nonzero eigenvector $\boldsymbol{x} \geqslant 0$.*
(iii) *$\rho(A)$ does not decrease when any entry of A is increased.*

Corollary 1.10 (Varga [10, Corollary 3.20]). *If $A = [a_{ij}]$ is a real, irreducibly diagonally dominant $n \times n$ matrix with $a_{i,j} \leqslant 0$ for all $i \neq j$, and $a_{i,i} > 0$ for all $1 \leqslant i \leqslant n$, then $A^{-1} > O$.*

Theorem 1.11 (Gunawardena et al. [3, Theorem 2.2]). *Let A be a nonnegative matrix. Then:*

(i) *If $\alpha\boldsymbol{x} \leqslant A\boldsymbol{x}$ for some nonnegative vector $\boldsymbol{x}$, $\boldsymbol{x} \neq 0$, then $\alpha \leqslant \rho(A)$.*
(ii) *If $A\boldsymbol{x} \leqslant \beta\boldsymbol{x}$ for some positive vector $\boldsymbol{x}$, then $\rho(A) \leqslant \beta$. Moreover, if A is irreducible and if*

$$0 \neq \alpha\boldsymbol{x} \leqslant A\boldsymbol{x} \leqslant \beta\boldsymbol{x}$$

for some nonnegative vector $\boldsymbol{x}$, then $\alpha \leqslant \rho(A) \leqslant \beta$ and $\boldsymbol{x}$ is a positive vector.

Lemma 1.12. *Let $B = L + U$ be a nonnegative irreducible $n \times n$ Jacobi matrix, $n \geqslant 2$. The spectral radius of the Gauss–Seidel matrix $T_{\mathrm{GS}} = (I - L)^{-1}U$ is positive and that its associated eigenvectors has positive components* (Varga [10, Exercises 3.3.6]).

Lemma 1.13 (Neumann and Plemmons [8, Lemma 2.2]). *Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splittings of the monotone matrices A_1* and *A_2, respectively, such that $M_2^{-1} \geqslant M_1^{-1}$. If there exists a positive vector $\boldsymbol{x}$ such that*

$$0 \leqslant A_1\boldsymbol{x} \leqslant A_2\boldsymbol{x},$$

then for the monotonic norm Neumann and Plemmons, [9] *associated with $\boldsymbol{x}$,*

$$\|M_2^{-1}N_2\|_x \leqslant \|M_1^{-1}N_1\|_x.$$

In particular, if $M_1^{-1}N_1$ has a positive perron vector, then

$$\rho(M_2^{-1}N_2) \leqslant \rho(M_1^{-1}N_1).$$

2. Comparison theorem

By using the results of previous section, we should prove comparison theorems between preconditioned iterative methods which are presented in Section 1.

Before proving the comparison theorems, we discuss about irreducibility of the preconditioned matrices. First, we consider A_C for the following matrix. For

$$A^{(0)} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix},$$

we obtained

$$A_C^{(0)} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{4} & -\frac{7}{12} \\ 0 & -\frac{1}{2} & \frac{5}{6} \end{pmatrix}.$$

Clearly, $A_C^{(0)}$ is always reducible. Next, we consider about irreducibly for A_S and $A_{\max}$. For irreducibly diagonally dominant Z-matrix A with order 2, A_S and $A_{\max}$ are always reducible. We next consider matrices with order 3 having the same condition as follows:

Example 2.1. Put $A^{(k)}$ is irreducible, where $k = 1, 2, 3$. For

$$A^{(1)} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{3} & 1 & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix},$$

we have

$$A_S^{(1)} = A_{\max}^{(1)} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{4}{9} & \frac{8}{9} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}.$$

Clearly, $A_S^{(1)}$ and $A_{\max}^{(1)}$ are irreducible. For

$$A^{(2)} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix},$$

we have

$$A_S^{(2)} = A_{\max}^{(2)} = \begin{pmatrix} 1 & -1 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

Since there is no path from P_2 to P_3, both $A_S^{(2)}$ and $A_{\max}^{(2)}$ are reducible. As same way,

$$A^{(3)} = \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{2} \\ 0 & 1 & -1 \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}, \quad A_S^{(3)} = \begin{pmatrix} 1 & 0 & -\frac{5}{6} \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}, \quad A_{\max}^{(3)} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}.$$

$A_S^{(3)}$ is irreducible but $A_{\max}^{(3)}$ is reducible, because there is no path from P_1 to P_3. As known example, when A is irreducible, A_α, A_S and $A_{\max}$ are irreducible or reducible. Let A be irreducible. By putting $\alpha \neq 1$ in (5), we have $\bar{a}_{ii+1} \neq 0$, and therefore $A_{\alpha\neq 1}$ has same zero entries as A is thus irreducible.

Lastly, we consider about diagonal dominance for the preconditioned matrices. Kohno et al. [4] introduced useful theorem as follows.

Theorem 2.2. *Let A be a nonsingular diagonally dominant Z-matrix. Then A_α is a diagonally dominant Z-matrix, and*

$$\rho(T_{\alpha<1}) < 1 \quad \textit{for} \;\; 0 \leqslant \alpha_i \leqslant 1 \;\; (1 \leqslant i \leqslant n).$$

Proof. See [4, Theorem 3].

As shown Example 2.1, A_α satisfies Theorem 2.2. Based on the same arguments as the Theorem 2.2, we obtain that A_C and $A_{\max}$ are also a diagonally dominant Z-matrix. □

Lemma 2.3. *Let A be an irreducibly diagonally dominant Z-matrix, where $A = M - N$ and $A_\alpha = (I + \alpha S)A = M_\alpha - N_\alpha$ are Gauss–Seidel splittings. Then the following inequality holds,*

$$M_{\alpha>1}^{-1} \geqslant M_{\alpha=1}^{-1} \geqslant M_{\alpha<1}^{-1} \geqslant M^{-1} \geqslant O. \tag{7}$$

Proof. Since $L \geqslant O$, it easily follows that,

$$M^{-1} = (I - L)^{-1} = (I + L + L^2 + \cdots + L^{n-1}) \geqslant O.$$

Diagonal elements of the preconditioned matrix A_α is as follows:

$$1 - \alpha a_{ii+1} a_{i+1i}, \quad \text{for } 1 \leqslant i < n.$$

By putting

$$\alpha' = \min_{1 \leqslant i < n} (1 - \alpha a_{ii+1} a_{i+1i}) > 0,$$

we have $(I - \alpha D_S) > O$ for $\alpha' \geqslant \alpha > 1$ ([4, Theorem 4]).

Clearly, $(L + \alpha E_S) \geqslant O$, then the following inequality holds:

$$\begin{aligned} M_\alpha^{-1} = {} & [I + \{(I - \alpha D_S)^{-1}(L + \alpha E_S)\} + \{(I - \alpha D_S)^{-1}(L + \alpha E_S)\}^2 + \cdots \\ & + \{(I - \alpha D_S)^{-1}(L + \alpha E_S)\}^{n-1}](I - \alpha D_S)^{-1} \geqslant O. \end{aligned}$$

In the case $0 < \alpha \leqslant 1$, $(I - \alpha D_S)^{-1} \geqslant I$ and $L + \alpha E_S \geqslant L$ hold. Thus we have

$$M_{\alpha=1}^{-1} \geqslant M_{\alpha<1}^{-1} \geqslant M^{-1} \geqslant O. \tag{8}$$

In the case $\alpha \geqslant 1$, since we have $(I - \alpha D_S)^{-1} \geqslant (I - D_S)^{-1}$, and $(L + \alpha E_S) \geqslant (L + E_S) \geqslant O$, then the following inequality holds:

$$M_{\alpha>1}^{-1} \geqslant M_{\alpha=1}^{-1}. \tag{9}$$

From above results, we have

$$M_{\alpha>1}^{-1} \geqslant M_{\alpha=1}^{-1} \geqslant M_{\alpha<1}^{-1} \geqslant M^{-1} \geqslant O. \qquad \square$$

Theorem 2.4. *Let A be an irreducibly diagonally dominant Z-matrix, where $A = M - N$* and *$A_{\alpha<1} = (I + \alpha S)A = M_{\alpha<1} - N_{\alpha<1}$ are Gauss–Seidel regular splittings. Then the following relation holds,*

$$\rho(M_{\alpha<1}^{-1} N_{\alpha<1}) \leqslant \rho(M^{-1}N) < 1. \tag{10}$$

Proof. From Definition 1.2 and Theorem 1.7, there exists $\boldsymbol{x} \geqslant 0$ such that $A\boldsymbol{x} \geqslant 0$. Since $\alpha S \geqslant O$, the following inequality holds:

$$A_{\alpha<1}(A^{-1} - A_{\alpha<1}^{-1})A\boldsymbol{x} = (A_{\alpha<1} - A)\boldsymbol{x} = \{(I + \alpha S)A - A\}\boldsymbol{x} = \alpha SA\boldsymbol{x} \geqslant 0. \tag{11}$$

From assumption and Theorem 2.2, A and $A_{\alpha<1}$ are diagonally dominant Z-matrices. Therefore, from Corollary 1.9, A and $A_{\alpha<1}$ are M-matrices. So in (11), $A^{-1} \geqslant A_{\alpha<1}^{-1}$ holds. From Remark 1.2(i), $N \geqslant N_{\alpha<1} \geqslant O$. From Lemma 1.12, $M^{-1}N$ has the positive vector $\boldsymbol{y}$. Then it follows that

$$\begin{aligned} A^{-1}N\boldsymbol{y} &= (I - M^{-1}N)^{-1}M^{-1}N\boldsymbol{y} \\ &= \frac{\rho(M^{-1}N)}{1 - \rho(M^{-1}N)}\boldsymbol{y} \\ &\geqslant (I - M_{\alpha<1}^{-1}N_{\alpha<1})^{-1}M_{\alpha<1}N_{\alpha<1}\boldsymbol{y}, \end{aligned}$$

which by Theorem 1.11 implies

$$\frac{\rho(M^{-1}N)}{1 - \rho(M^{-1}N)} \geqslant \frac{\rho(M_{\alpha<1}^{-1}N_{\alpha<1})}{1 - \rho(M_{\alpha<1}^{-1}N_{\alpha<1})}.$$

The monotonicity of the function $f(\alpha) = \alpha/(1 - \alpha)$ implies

$$1 > \rho(M^{-1}N) \geqslant \rho(M_{\alpha<1}^{-1}N). \qquad \square$$

From Definition 1.3 and Corollary 1.10, an irreducibly diagonally dominant Z-matrix is an M-matrix. Then here after we use terminology "M-matrix" instead of irreducibly diagonally dominant Z-matrix.

Theorem 2.5. *Let $A = M - N$ be an M-matrix, and $A_{\alpha=1} = M_{\alpha=1} - N_{\alpha=1}$ and $A_{\alpha<1} = M_{\alpha<1} - N_{\alpha<1}$. Then the following inequality holds,*

$$\rho(M_{\alpha<1}^{-1}N_{\alpha<1}) \geqslant \rho(M_{\alpha=1}^{-1}N_{\alpha=1}).$$

Proof. From Theorem 2.2, as $A_{\alpha=1}$ is an M-matrix, $A_{\alpha=1}^{-1} \geqslant O$ holds. As same ways as the proof of Theorem 2.4, we have $A_{\alpha<1}^{-1} \geqslant A_{\alpha=1}^{-1} \geqslant O$. It is follows from Remark 1.2(i) and (8) that $N_{\alpha<1} \geqslant N_{\alpha=1}$ and $M_{\alpha=1}^{-1} \geqslant M_{\alpha<1}^{-1} \geqslant O$. Thus, by the arguments as Theorem 2.6, the inequality $\rho(M_{\alpha<1}^{-1}N_{\alpha<1}) \geqslant \rho(M_{\alpha=1}^{-1}N_{\alpha=1})$ holds. □

Theorem 2.6. *Let A be an M-matrix. Then $A_{\alpha>1} = (I + \alpha S)A = M_{\alpha>1} - N_{\alpha>1}$ is the weak regular splitting. Then*

$$\rho(M_{\alpha>1}^{-1}N_{\alpha>1}) \leqslant \rho(M_{\alpha=1}^{-1}N_{\alpha=1}) < 1.$$

Proof. $A_{\alpha>1}$ is an irreducibly diagonally dominant matrix. From Remark 1.1(ii), $A_{\alpha>1}$ is a weak regular splitting. From Theorem 1.8 there exists $\boldsymbol{x} \geqslant \boldsymbol{0}$ such that

$$M_{\alpha>1}^{-1}N_{\alpha>1}\boldsymbol{x} = \rho(M_{\alpha>1}^{-1}N_{\alpha>1})\boldsymbol{x}.$$

Since A is an M-matrix, $A\boldsymbol{x} \geqslant \boldsymbol{0}$ holds, and the following inequality holds:

$$(A_{\alpha>1} - A_{\alpha=1})\boldsymbol{x} = (\alpha - 1)SA\boldsymbol{x} \geqslant \boldsymbol{0}.$$

Then the following relation holds:

$$\begin{aligned} M_{\alpha>1}^{-1}N_{\alpha>1}\boldsymbol{x} &= (I - M_{\alpha>1}^{-1}N_{\alpha>1})\boldsymbol{x} \\ &\geqslant M_{\alpha=1}^{-1}N_{\alpha=1}\boldsymbol{x} \\ &= (I - M_{\alpha=1}^{-1}N_{\alpha=1})\boldsymbol{x} \end{aligned}$$

which Theorem 1.11 implies

$$\rho(M_{\alpha>1}^{-1}N_{\alpha>1}) \leqslant \rho(M_{\alpha=1}^{-1}N_{\alpha=1}) < 1. \qquad \square$$

Lemma 2.7 (Kotakemori et al. [5, Lemma 3.4]). *Let A be an M-matrix. Suppose that $a_{ii+1}a_{i+1j} \leqslant a_{ik_i}a_{k_ij}$, $1 \leqslant i < n-1$, $j < i$. Put $A_{\max} = M_{\max} - N_{\max}$ is the Gauss–Seidel splitting. Then the following inequality holds*:

$$M_{\max}^{-1} \geqslant M_{\alpha=1}^{-1} \geqslant O.$$

Proof. From assumptions, $E_{\max} \geqslant E_S$ and $D_{\max} \geqslant D_S$ hold, then it follows that:

$$M_{\max}^{-1} \geqslant M_{\alpha=1}^{-1} \geqslant O. \qquad \square$$

Theorem 2.8. *Let A be an M-matrix. Then* $A_{\alpha=1} = M_{\alpha=1} - N_{\alpha=1}$ *and* $A_{\max} = M_{\max} - N_{\max}$ *are Gauss–Seidel regular splittings. Under the assumption in Lemma* 2.7, *there exists a positive vector* $\boldsymbol{x}$ *such that* $A_{\max}\boldsymbol{x} \geqslant A_{\alpha=1}\boldsymbol{x} \geqslant 0$. *Then*

$$\rho(M_{\max}^{-1}N_{\max}) \leqslant \rho(M_{\alpha=1}^{-1}N_{\alpha=1}).$$

Proof. Consider any vector $\boldsymbol{e} > 0$ (e.g., with all component equal to 1), and $\boldsymbol{x} = A^{-1}\boldsymbol{e}$. No row of A^{-1} can have all null entries, then $\boldsymbol{x} > 0$. Then the following equation holds:

$$(A_{\max} - A_{\alpha=1})\boldsymbol{x} = (S_{\max} - S)A\boldsymbol{x} = (S_{\max} - S)\boldsymbol{e} \geqslant 0.$$

Since $M_{\max}^{-1} \geqslant M_{\alpha=1}^{-1} \geqslant O$, we have

$$M_{\max}^{-1}A_{\max}\boldsymbol{x} = (I - M_{\max}^{-1}N_{\max})\boldsymbol{x} \geqslant (M_{\alpha=1}^{-1}A_{\alpha=1})\boldsymbol{x} = (I - M_{\alpha=1}^{-1}N_{\alpha=1})\boldsymbol{x}. \tag{12}$$

Thus, it follows that

$$\|M_{\max}^{-1}N_{\max}\|_x \leqslant \|M_{\alpha=1}^{-1}N_{\alpha=1}\|_x.$$

Since $A_{\alpha=1} = M_{\alpha=1} - N_{\alpha=1}$ is the Gauss–Seidel convergent regular splitting and Lemma 1.12, $M_s^{-1}N_s$ has the positive vector $\boldsymbol{y}$. Therefore the following inequality holds:

$$\rho(M_{\max}^{-1}N_{\max}) \leqslant \rho(M_{\alpha=1}^{-1}N_{\alpha=1}) \leqslant \rho(M^{-1}N) < 1. \qquad \square$$

Theorem 2.9. *Let A be an M-matrix. Then* $A_{\alpha=1} = M_{\alpha=1} - N_{\alpha=1}$ *and* $A_R = (I + S + R)A = M_R - N_R$ *are Gauss–Seidel regular splittings. Then the following inequality holds,*

$$\rho(M_R^{-1}N_R) \leqslant \rho(M_{\alpha=1}^{-1}N_{\alpha=1}) < 1.$$

Proof. As same way Theorem 2.3, there exists $\boldsymbol{x} \geqslant \boldsymbol{0}$ such that $A\boldsymbol{x} \geqslant \boldsymbol{0}$. Since $R \geqslant O$, the following inequality holds:

$$\begin{aligned}(A_R - A_S)\boldsymbol{x} &= \{(I + S + R) - (I + R)\}A\boldsymbol{x}\\ &= RA\boldsymbol{x} \geqslant \boldsymbol{0}.\end{aligned}$$

So, the following relation holds:

$$A_R(A_S^{-1} - A_R^{-1})A_S\boldsymbol{x} = (A_R - A_S)\boldsymbol{x} \geqslant \boldsymbol{0}.$$

Since A_α and A_R are M-matrix, we have

$$A_S^{-1} \geqslant A_R^{-1} \geqslant O.$$

By noting that $N_S = N_R$ as described Remark 1.2(ii), the following relation holds:

$$\begin{aligned}A_S^{-1}N_S\boldsymbol{x} &= (I - M_S^{-1}N_S)M_S^{-1}N_S\boldsymbol{x}\\ &= \frac{\rho(M_S^{-1}N_S)}{1 - \rho(M_S^{-1}N_S)}\boldsymbol{x}\\ &\geqslant (I - M_R^{-1}N_S)^{-1}M_R^{-1}N_S\boldsymbol{x} \geqslant \boldsymbol{0}\end{aligned}$$

which by Theorem 1.9 implies

$$\frac{\rho(M_S^{-1}N_S)}{1-\rho(M_S^{-1}N_S)} \geqslant \frac{\rho(M_R^{-1}N_S)}{1-\rho(M_R^{-1}N_S)}.$$

The monotonicity of the function $f(\alpha)=\alpha/(1-\alpha)$ implies

$$1 > \rho(M_S^{-1}N_S) \geqslant \rho(M_R^{-1}N_R). \qquad \square$$

Remark 2.10. In the above theorem, if we replace operators A_α and $A_R=(I+S+R)A$ by $A_{\max}$ and $A_R=(I+S_{\max}+R)A$, respectively, then $\rho(M_R^{-1}N_R) \leqslant \rho(M_{\max}^{-1}N_{\max}) < 1$ holds. As same way, for rest preconditioners we have same results.

3. Numerical examples

In this section, we compare preconditioned Gauss–Sidel methods for the following the problems. It is very difficult work to obtain both optimum α_i and optimum β_i by numerical computation. So, we use only one optimum α and β by numerical experiment.

C1: the linear equations systems arising from the application of five point finite difference scheme to the Laplace equation in a square region. The boundary conditions are given as follows:

$$u(x,0)=1 \quad \text{on} \quad 0 \leqslant x \leqslant 1,$$
$$u(x,y)=0 \quad \text{on} \quad \text{other boundary}.$$

We adopted $\boldsymbol{u}^{(0)}=\mathbf{0}$ as initial guess. Let the convergence criterion be $\|u(x,y)^{(k+1)}-u(x,y)^{(k)}\|_2/\|u(x,y)^{(k+1)}\|_2 \leqslant 10^{-12}$. The numerical results are shown in Tables 1 and 2. The CPU time for computing α_{opt} and β_{opt} is neglected.

C2: the irreducibly diagonally dominant matrix is generated randomly.

This is the case that the coefficient matrix of the linear equations system is dense. Numerical results are shown in Table 3.

Table 1
Comparison of spectral radii of iteration matrices

n	Preconditioner								
	I	$I+C$	$I+S$	$I+\alpha S$	(α_{opt})	$I+\beta U$	(β_{opt})	$I+S_{\max}$	$I+S+R$
4	0.2500	0.0400	0.1111	0.0738	(1.1)	0.0167	(1.1)	0.0927	0.0920
16	0.6545	0.6483	0.4999	0.2775	(1.6)	0.1339	(1.4)	0.4904	0.4986
64	0.8830	0.8829	0.8135	0.4880	(2.4)	0.3225	(1.8)	0.8126	0.8134
256	0.9662	0.9662	0.9444	0.7075	(3.1)	0.5879	(2.2)	0.9444	0.9444

Table 2
Comparison of (a) iteration numbers and (b) CPU time

n	Preconditioner						
	I	$I+C$	$I+S$	$I+\alpha S$	$I+\beta U$	$I+S_{\max}$	$I+S+R$
(a) *Iteration time*							
4	21	16	13	11	7	12	12
16	63	62	39	21	14	38	39
64	207	206	124	36	24	124	124
256	719	719	431	65	52	431	431
1024	2614	2613	1567	120	152	1566	1567
(b) *CPU time*							
4	0.89	3.39	3.25	3.25	3.09	3.25	3.27
16	8.72	13.88	11.25	8.45	7.47	11.19	11.30
64	103.81	124.70	89.44	33.89	29.09	91.08	89.75
256	15.47	17.61	12.59	2.25	2.19	12.75	12.58
1024	216.45	258.58	187.70	18.72	26.25	177.61	185.55

Table 3
Comparison of spectral radii

n	Preconditioner								
	I	$I+C$	$I+S$	$I+\alpha S$	(α_{opt})	$I+\beta U$	(β_{opt})	$I+S_{\max}$	$I+S+R$
10	0.3944	0.3553	0.3515	0.1664	(3.8)	0.0464	(1.8)	0.3475	0.3197
50	0.4977	0.4913	0.4855	0.2527	(19.1)	0.0562	(1.9)	0.4811	0.4781
100	0.5004	0.4978	0.4939	0.2784	(34.5)	0.0574	(1.9)	0.4929	0.4905
150	0.5107	0.5086	0.5065	0.2521	(50.5)	0.0677	(2.0)	0.5050	0.5049

For the case that $n = 10$, the coefficient matrix

$$\begin{bmatrix} 1 & -0.0510 & -0.1353 & -0.0659 & -0.0030 & -0.0838 & -0.0143 & -0.0730 & -0.1566 & -0.1132 \\ -0.0976 & 1 & -0.0303 & -0.0693 & -0.0717 & -0.0895 & -0.0166 & -0.0485 & -0.0422 & -0.1600 \\ -0.1163 & -0.0294 & 1 & -0.0493 & -0.0152 & -0.1050 & -0.1104 & -0.0927 & -0.0744 & -0.0189 \\ -0.0097 & -0.0031 & -0.0951 & 1 & -0.0098 & -0.1615 & -0.1279 & -0.1484 & -0.0671 & -0.1749 \\ -0.0421 & -0.1120 & -0.0842 & -0.1057 & 1 & -0.0030 & -0.0132 & -0.0211 & -0.0670 & -0.1046 \\ -0.1212 & -0.0009 & -0.0243 & -0.0916 & -0.0923 & 1 & -0.0353 & -0.0744 & -0.0177 & -0.0869 \\ -0.0205 & -0.0027 & -0.1342 & -0.0831 & -0.2819 & -0.1193 & 1 & -0.0709 & -0.0250 & -0.0240 \\ -0.0853 & -0.0018 & -0.1065 & -0.1581 & -0.0868 & -0.1004 & -0.0718 & 1 & -0.0472 & -0.0597 \\ -0.0299 & -0.0416 & -0.0366 & -0.0157 & -0.0753 & -0.0819 & -0.0735 & -0.0856 & 1 & -0.0783 \\ -0.0792 & -0.0553 & -0.0597 & -0.0103 & -0.0718 & -0.1030 & -0.1261 & -0.0107 & -0.0746 & 1 \end{bmatrix}.$$

4. Conclusion

From comparison theorems and numerical experiments, it is may be concluded that preconditioners are effective to accelerate convergence of the Gauss–Sidel method. The preconditioners $I+\alpha S$, $I+\beta U$ give us excellent effects to improve the rate of convergence if we could determine α_{opt} and β_{opt}. A development of estimation formula for optimum α and optimum β remain future research.

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